

ESTIMATION OF NETWORK STRUCTURES FROM PARTIALLY OBSERVED MARKOV RANDOM FIELDS

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ABSTRACT. We consider the estimation of high-dimensional network structures from partially observed Markov random field data using a penalized pseudo-likelihood approach. We fit a misspecified model obtained by ignoring the missing data problem. We study the consistency of the estimator and derive a bound on its rate of convergence. The results obtained relate the rate of convergence of the estimator to the extent of the missing data problem. We report some simulation results that empirically validate some of the theoretical findings.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The problem of high-dimensional network structure estimation has recently attracted a lot of attention in statistics and machine learning. Both in the continuous case using Gaussian graphical models (Drton and Perlman (2004); Meinshausen and Bühlmann (2006); Yuan and Lin (2007); d’Aspremont et al. (2008); Bickel and Levina (2008); Rothman et al. (2008); Lam and Fan (2009)), and in the discrete case using Markov random fields (Banerjee et al. (2008); Höfling and Tibshirani (2009); Ravikumar et al. (2010); Guo et al. (2010)). This paper focuses mainly on Markov Random Fields (MRF) for non-Gaussian data. The problem can be described as follows. Let $(X^{(1)}, \dots, X^{(n)})$ be n i.i.d. random variables where $X^{(i)} = (X_1^{(i)}, \dots, X_p^{(i)})$ is a p -dimensional vector of dependent random variables with joint density

$$f_\theta(x_1, \dots, x_p) = \frac{1}{Z_\theta} \exp \left\{ \sum_{s=1}^p (A(x_s) + \theta(s, s)B_0(x_s)) + \sum_{1 \leq s < s' \leq p} \theta(s, s')B(x_s, x_{s'}) \right\}, \quad (1)$$

for known functions $A, B_0 : \mathsf{X} \rightarrow \mathbb{R}$ and a symmetric function $B : \mathsf{X} \times \mathsf{X} \rightarrow \mathbb{R}$, where X is a compact (generally finite) set. The real-valued symmetric matrix $\theta = \{\theta(s, s'), 1 \leq s, s' \leq p\}$ is the network structure and is the parameter of interest. The term Z_θ is a normalizing constant. This type of statistical models was pioneered by J. Besag (Besag (1974)) under the name of auto-model and we adopt the same name here, although Besag’s

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auto-models corresponds to setting $B(x, y) = xy$ above. The nice feature of model (1) is that for any $1 \leq s \leq p$, the conditional density of X_s given $\{X_j, j \neq s\} = x \in \mathbf{X}^{p-1}$ is

$$f_{\theta}^{(s)}(u|x) = \frac{1}{Z_{\theta}^{(s)}} \exp \left\{ A(u) + \theta(s, s)B_0(u) + \sum_{j \neq i} \theta(s, j)B(u, x_j) \right\}, \quad (2)$$

for a normalizing constant $Z_{\theta}^{(s)} = Z_{\theta}^{(s)}(x)$. Therefore, $\theta(s, j) = 0$ implies that X_s and X_j are conditionally independent given the other variables X_k , $k \notin \{s, j\}$. Thus estimating θ provides us with the dependence structure and the magnitude of the dependence between these variables.

This paper focuses on the situation where the outcomes $X_j^{(i)}$ are either categorical (\mathbf{X} is a finite set) or continuous bounded ($\mathbf{X} \subset \mathbb{R}^{m \times n}$ is compact). Based on $(X^{(1)}, \dots, X^{(n)})$, the true network structure denoted $\theta_{\star} = \{\theta_{\star}(s, s'), 1 \leq s, s' \leq p\}$ can be consistently estimated using a number of methods, even when the number of entries of θ_{\star} is much larger than n (Höfling and Tibshirani (2009); Ravikumar et al. (2010); Guo et al. (2010)). For computational tractability, a pseudo-likelihood approach is often preferred, even though it incurs a certain loss of efficiency. In the case of the auto-logistic model (where $\mathbf{X} = \{0, 1\}$, $A_0(u) = 0$, $B_0(u) = u$, $B(u, v) = uv$), Guo et al. (2010) shows that the ℓ^1 -penalized pseudo-likelihood estimator of θ_{\star} is consistent with ℓ^2 rate of convergence bounded from above by $\alpha^{-1} \sqrt{a \log p/n}$, where a is the number of non-zero elements of θ_{\star} and α is the smallest eigenvalue of the information matrix. Ravikumar et al. (2010) obtained similar results for a one-neighborhood-at-the-time ℓ^1 -penalized pseudo-likelihood estimator. Xue et al. (2010) also derived some properties of the oracle estimator with the SCAD penalty.

In many situations where network estimation is needed, the network data is only partially observed because certain nodes are missing from the sample. For example, in social network analysis, some close friends or siblings might not be part of the survey. As another example, in protein-protein networks, the analysis is often restricted to the specific subgroup of proteins that is believed to carry a role in a given biological function. So doing, some important but not yet identified proteins might be omitted from the analysis. This paper considers the problem of network estimation from partially observed MRF data. The issue cannot be completely addressed by simply ignoring the missing nodes and assuming that the observed data follows a MRF. This is because, unlike Gaussian distributions, Markov Random Field distributions are not closed under marginalization. For example, if there exist r additional nodes denoted $p+1, \dots, p+r$ such that the joint distribution of $(X_1, \dots, X_p, X_{p+1}, \dots, X_{p+r})$ is an auto-model with network structure $\{\theta(s, s'), 1 \leq s, s' \leq p+r\}$, then the joint (marginal) distribution of (X_1, \dots, X_p) is not of the form (1) in general. To take a specific example, if $r = 1$ and $A = B_0 \equiv 0$ and $B(x, y) = B(x)B(y)$, then the joint (marginal) distribution of (X_1, \dots, X_p) is the mixture

distribution

$$f_{\theta}(x_1, \dots, x_p) = Z_{\theta}^{-1} \sum_{i \in \mathbf{X}} \exp \left\{ \sum_{s=1}^p \theta_i(s) B(x_s) + \sum_{1 \leq s < s' \leq p} \theta(s, s') B(x_s) B(x_{s'}) \right\},$$

where $\theta_i(s) = B(i)\theta(s, p+1)$. Furthermore, the conditional distributions are altered. Indeed, and keeping with the assumption $r = 1$, if $|\theta(s, p+1)| > 0$, then the conditional density of X_s given $\{X_{\ell}, \ell \neq s, 1 \leq \ell \leq p\}$ depends not only X_{ℓ} for all ℓ such that $|\theta(s, \ell)| > 0$, but also on X_k for all k such that $|\theta(k, p+1)| > 0$. However, if $\theta(s, p+1) = 0$, the conditional density of X_s given $\{X_{\ell}, \ell \neq s, 1 \leq \ell \leq p\}$ remains (2). This suggests that if we ignore the missing nodes and fit the misspecified model (1) to the observed data, the resulting estimator will be well-behaved to the extent that the missing data problem is limited. That is, to the extent that $\sum_{s=1}^p |\theta_{\star}(s, p+1)|$ is small in the case $r = 1$ considered above.

The goal of the paper is to formalize this idea. In order to do so, we consider an infinite-volume Markov random field model, where only part of the field is observed, and we fit the misspecified model (1) using penalized pseudo-likelihood approach. We derive a general consistency result and show that under certain conditions, the estimators converges at the rate of $(\sqrt{a_n \log p_n/n} + \tau_n b_n)/\alpha_n$, where p_n is the number of observed nodes, a_n is the number of non-zero entries of the true network, α_n is the smallest eigenvalue of the Fisher information matrix, and where the term $\tau_n b_n$ quantifies the effect of the missing nodes (see Theorem 1.4 for a more rigorous statement). We conclude that the estimator $\hat{\theta}_n$ is robust to a small to moderate amount of missing data. We report some simulation results that are consistent with these findings. In practical situations where MRF are used, it is often unclear whether one is dealing with a partially observed field with important missing nodes. The above discussion thus stresses the need for methods of detecting the existence of missing nodes in Markov random field data. We leave this problem for future research, as it requires a better understanding of the asymptotic behavior of $\hat{\theta}_n$.

The paper is organized as follows. The infinite-volume Markov random field setting and the estimators are presented in Section 1.1. The paper presents two main results: Theorem 1.2 (and Corollary 1.3) on the consistency of the estimator, and Theorem 1.4 (and Corollary 1.5) on its rate of convergence. These results are presented in Section 1.2. The simulation example is presented in Section 1.3. Section 2 develops the technical proofs.

1.1. The setting. Let $(\mathbf{X}, \mathcal{E}, \rho)$ be a measure space. We assume that \mathbf{X} is a compact subset of $\mathbb{R}^{m_{\mathbf{X}}}$, \mathcal{E} its Borel sigma-algebra, and ρ a finite measure. The compactness of \mathbf{X} is wrt the usual Euclidean metric. \mathbf{X} is the sample space of the observations X_i . The main case of interest is the case where \mathbf{X} is finite. Let \mathcal{S} be a countably infinite set (typically, \mathcal{S} is a subset of the Euclidean space $\mathbb{R}^{m_{\mathcal{S}}}$ for some finite integer $m_{\mathcal{S}} \geq 1$). The set \mathcal{S} represents the nodes of the network. We assume that \mathcal{S} is equipped with

a linear ordering \succeq (for example, the lexicographical ordering of \mathbb{R}^{m_S}). We introduce $\underline{\mathcal{S}}^2 \stackrel{\text{def}}{=} \{(s, \ell) \in \mathcal{S} \times \mathcal{S} : \ell \succeq s\}$, the set of all ordered pairs of \mathcal{S} . More generally, if Λ is a subset of \mathcal{S} , we denote by $\underline{\Lambda}^2$, the set of all ordered pairs $(u, v) \in \Lambda \times \Lambda$, with $v \succeq u$.

Let $A, B_0 : \mathsf{X} \rightarrow \mathbb{R}$, $B : \mathsf{X} \times \mathsf{X} \rightarrow \mathbb{R}$ be known measurable functions such that $B(x, y) = B(y, x)$ (symmetry). We also assume that the diagonal of B is B_0 : $B(x, x) = B_0(x)$ for all $x \in \mathsf{X}$. We assume throughout the paper that

$$\|A\|_\infty < \infty, \quad \|B_0\|_\infty < \infty, \quad \text{and} \quad \|B\|_\infty < \infty. \quad (3)$$

In the above, $\|f\|_\infty$ is the supremum norm.

An infinite matrix is a map from $\mathcal{S} \times \mathcal{S}$ to \mathbb{R} . For an infinite matrix $\theta : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ and $s \in \mathcal{S}$, the θ -neighborhood of s is the set

$$\partial_\theta s \stackrel{\text{def}}{=} \{\ell \in \mathcal{S} : \ell \neq s \text{ and } |\theta(s, \ell)| > 0\},$$

and the θ -degree of node s is the quantity (possibly infinite)

$$\deg(s, \theta) \stackrel{\text{def}}{=} \sum_{\ell \in \mathcal{S} \setminus \{s\}} |\theta(s, \ell)| = \sum_{\ell \in \partial_\theta s} |\theta(s, \ell)|.$$

We denote \mathcal{M} the space of all infinite symmetric matrices θ such that $\deg(s, \theta) < \infty$ for all $s \in \mathcal{S}$. For $q \in [1, \infty)$, we denote by \mathcal{M}_q the Banach space of all infinite symmetric matrices $\theta \in \mathcal{M}$ such that

$$\|\theta\|_q \stackrel{\text{def}}{=} \left\{ \sum_{(s, \ell) \in \underline{\mathcal{S}}^2} |\theta(s, \ell)|^q \right\}^{1/q} < \infty.$$

Let $(\Omega, \mathcal{F}) = (\mathsf{X}^{\mathcal{S}}, \mathcal{E}^{\mathcal{S}})$ be the product space equipped with the product topology and its Borel sigma-algebra. For $\theta \in \mathcal{M}$, let μ_θ be the probability measure on (Ω, \mathcal{F}) such that if $\{X_s, s \in \mathcal{S}\}$ is a stochastic process with distribution μ_θ , the conditional distribution of X_s given the sigma-algebra generated by $\{X_\ell, \ell \neq s\}$ exists and has density (wrt ρ) $f_\theta^{(s)}(\cdot|x)$, where for $u \in \mathsf{X}$, $x \in \mathsf{X}^{\mathcal{S} \setminus \{s\}}$,

$$f_\theta^{(s)}(u|x) = \frac{1}{Z_\theta^{(s)}} \exp \left\{ A(u) + \theta(s, s)B_0(u) + \sum_{\ell \in \mathcal{S} \setminus \{s\}} \theta(s, \ell)B(u, x_\ell) \right\}, \quad (4)$$

for a normalizing constant $Z_\theta^{(s)}$. Notice that $f_\theta^{(s)}(u|x)$ actually depends only on $x_{\partial_\theta s} \stackrel{\text{def}}{=} \{x_\ell : \ell \in \partial_\theta s\}$. Under (3) and for $\theta \in \mathcal{M}$, such distribution μ_θ exists (but might not be unique in general). We refer the reader to Appendix 1 for a precise definition and existence of μ_θ . A random process $\{X_s, s \in \mathcal{S}\}$ with distribution μ_θ is called an infinite-volume auto-model random field. We denote by \mathbb{E}_θ the expectation operator with respect to μ_θ on (Ω, \mathcal{F}) . When θ is the true network structure θ_\star (introduced below), we simply write \mathbb{E}_\star instead of $\mathbb{E}_{\theta_\star}$. For $\Lambda \subseteq \mathcal{S}$, we denote X_Λ the stochastic process $\{X_s, s \in \Lambda\}$.

From $f_\theta^{(s)}$, and for a measurable function $H : \mathbf{X} \times \mathbf{X}^{\mathcal{S} \setminus \{s\}} \rightarrow \mathbb{R}$, we can obtain the conditional expectation $\mathbb{E}_\theta (H(X_s, X_{\mathcal{S} \setminus \{s\}}) | X_{\mathcal{S} \setminus \{s\}})$ as $\int_{\mathbf{X}} H(u, X_{\mathcal{S} \setminus \{s\}}) f_\theta^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) du$, provided the integral is well defined. And we can define similarly the conditional variance $\text{Var}_\theta (H(X_s, X_{\mathcal{S} \setminus \{s\}}) | X_{\mathcal{S} \setminus \{s\}})$.

For $\theta_\star \in \mathcal{M}$, let $\{X^{(i)}, i \geq 1\}$ be a sequence of i.i.d. infinite-volume random fields with distribution μ_{θ_\star} defined on some probability space with probability measure $\check{\mathbb{P}}_\star$ and expectation operator $\check{\mathbb{E}}_\star$. Let $\{D_n, n \geq 1\}$ be a sequence of increasing finite subsets of \mathcal{S} such that $D_n \uparrow \mathcal{S}$. For a finite set A , $|A|$ denotes its cardinality and we set $p_n = |D_n|$. For $n \geq 1$, let $d_n = p_n(p_n + 1)/2$ and denote $\mathcal{M}^{(n)}$ the space of all symmetric finite matrices $\{\theta(s, \ell), s, \ell \in D_n\}$, that we identify with \mathbb{R}^{d_n} .

We assume that for some $n \geq 1$, we observe partially each of the random field $X^{(i)}$ ($1 \leq i \leq n$) over the domain D_n giving rise to observations $X_{D_n}^{(i)} = \{X_s^{(i)}, s \in D_n\}$. The remaining points $\mathcal{S} \setminus D_n$ are not known and the associated random variables $X_{\mathcal{S} \setminus D_n}$ are not observed. We are interested in estimating the infinite matrix θ_\star . For $s \in \mathcal{S}$, we define $\partial s = \partial_{\theta_\star} s$ and called it the (true) neighborhood of s . We also define $\partial_n s \stackrel{\text{def}}{=} D_n \setminus \{s\}$. Since the neighborhood system $\{\partial s, s \in \mathcal{S}\}$ is not known, we introduce the approximate full conditional distributions

$$f_\theta^{(s)}(u | x_{\partial_n s}) \stackrel{\text{def}}{=} \frac{1}{Z_{n,\theta}^{(s)}} \exp \left(A(u) + \theta(s, s)B_0(u) + \sum_{\ell \in \partial_n s} \theta(s, \ell)B(u, x_\ell) \right), \quad (5)$$

for some normalizing constant $Z_{n,\theta}^{(s)}$. For $\lambda \geq 0$, let $q_\lambda : [0, \infty) \rightarrow [0, \infty)$ a penalty function. We then define the functions

$$\bar{\ell}_n(\theta) \stackrel{\text{def}}{=} \sum_{i=1}^n \sum_{s \in D_n} \log f_\theta^{(s)}(X_s^{(i)} | X_{\partial_n s}^{(i)}), \quad \text{and} \quad Q_n(\theta) = \bar{\ell}_n(\theta) - \sum_{(s,\ell) \in D_n^2} q_{\lambda_n}(|\theta(s, \ell)|), \quad \theta \in \mathcal{M}^{(n)},$$

for some parameter $\lambda_n > 0$. We are mainly interested in convex penalty functions, particularly the ℓ^1 penalty for which $q_\lambda(x) = \lambda x$. But we develop much of the results under the general condition A1 below that applies in principle to non-convex penalties such as the SCAD penalty of Fan and Li (2001).

A1 For any $\lambda \geq 0$, $q_\lambda(0) = 0$, q_λ is right-continuous at 0 and differentiable on $(0, \infty)$ and

$$\sup_{\lambda > 0} \sup_{x > 0} |q'_\lambda(x)| / \lambda < \infty. \quad (6)$$

Finally, we define

$$\text{Argmax } Q_n \stackrel{\text{def}}{=} \{\theta \in \mathcal{M}^{(n)} : Q_n(\theta) = \sup_{\vartheta \in \mathcal{M}^{(n)}} Q_n(\vartheta)\},$$

and we call any element $\hat{\theta}_n$ of $\text{Argmax } Q_n$ a maximizer of Q_n , that is a penalized pseudo-likelihood estimator of θ_\star .

Remark 1. We want to stress the fact that the sets \mathcal{S} and D_n are purely conceptual and need not be known. This is because we have replaced the full conditional density (4) by the approximation (5) in which the neighborhood of s is $\partial_n s = D_n \setminus \{s\}$, and without any loss of generality we can replace D_n by $\{1, \dots, p_n\}$. As a result, the computation of $\hat{\theta}_n$ does not make use of \mathcal{S} and D_n . For instance, with the ℓ^1 penalty, one obtains the same ℓ^1 -penalized pseudo-likelihood estimator as in Höfling and Tibshirani (2009); Guo et al. (2010).

It is useful to have some simple conditions under which $\text{Argmax } Q_n$ is not empty.

Proposition 1.1. *Fix $n \geq 1$. Suppose that for any $s \in \mathcal{S}$, there exists a finite constant $c(s)$ such that for all $\theta \in \mathcal{M}^{(n)}$, all $u \in \mathbf{X}$ and for all $x_{\partial_n s} \in \mathbf{X}^{\partial_n s}$,*

$$f_{\theta}^{(s)}(u|x_{\partial_n s}) \leq c(s).$$

Suppose also that for any $\alpha, \lambda \geq 0$ the set $\{x \geq 0 : q_{\lambda}(x) \leq \alpha\}$ is bounded. Then $\text{Argmax } Q_n$ is non-empty.

Remark 2. The result is not always useful. It applies to the ℓ^1 penalty but not to the SCAD penalty. If \mathbf{X} is finite as in all the examples below, then $f_{\theta}^{(s)}(\cdot|x_{\partial_n s})$ is a finite probability mass function. Therefore the assumption of the proposition holds with $c(s) = 1$.

Proof. Fix a sample path $\omega \in \Pi$. Then Q_n is a continuous \mathbb{R} -valued function on $\mathcal{M}^{(n)}$. Denote 0 the null element of $\mathcal{M}^{(n)}$, and $r = Q_n(0)$. Then $\mathbf{L}_r \stackrel{\text{def}}{=} \{\theta \in \mathcal{M}^{(n)} : Q_n(\theta) \geq r\}$ is nonempty and closed by continuity of Q_n . Under the assumption of the proposition, if $\theta \in \mathbf{L}_r$, then for any $(s, \ell) \in \underline{D}_n^2$:

$$q_{\lambda_n}(|\theta(s, \ell)|) \leq \sum_{(s, \ell) \in \underline{D}_n^2} q_{\lambda_n}(|\theta(s, \ell)|) \leq n \sum_{s \in D_n} \log c(s) - r.$$

Thus \mathbf{L}_r is a compact subset of $\mathcal{M}^{(n)}$ and Q_n attains its maximum at $\hat{\theta}_n \in \mathbf{L}_r$. \square

1.2. Consistency and rate of convergence. Let \mathcal{M}_1 be the separable Banach space of all $\theta \in \mathcal{M}$ such that $\|\theta\|_1 \stackrel{\text{def}}{=} \sum_{(u, v) \in \mathcal{S}^2} |\theta(u, v)| < \infty$. We investigate the consistency of $\hat{\theta}_n$ as a random element of \mathcal{M}_1 under the following sparsity assumption.

A2 $\theta_{\star} \in \mathcal{M}$ and for any $s \in \mathcal{S}$, the θ_{\star} -neighborhood of s (that is, the set $\partial_{\theta_{\star}} s = \{\ell \in \mathcal{S} \setminus \{s\} : \theta_{\star}(s, \ell) \neq 0\}$) is a finite set.

A2 guarantees that for $\theta \in \mathcal{M}_1$, $\theta + \theta_{\star} \in \mathcal{M}$, so that the full conditional densities $f_{\theta + \theta_{\star}}^{(s)}(u|x_{\mathcal{S} \setminus \{s\}})$ are well defined. For two matrices $\theta, \theta' \in \mathcal{M}$, we write $\theta \cdot \theta'$ to denote the component-wise product. And if $\theta \in \mathcal{M}$, and $n \geq 1$, $\theta^{(n)}$ denotes the element of $\mathcal{M}^{(n)}$ such

that $\theta^{(n)}(u, v) = \theta(u, v)$ if $(u, v) \in D_n \times D_n$ (and $\theta^{(n)}(u, v) = 0$ otherwise). We introduce

$$U_n(\theta) \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^n \sum_{s \in D_n} \left(\log f_{\theta_*}^{(s)}(X_s^{(i)} | X_{\partial_n s}^{(i)}) - \log f_{\theta_* + \theta}^{(s)}(X_s^{(i)} | X_{\partial_n s}^{(i)}) \right) \\ + n^{-1} \sum_{(s, \ell) \in \underline{D}_n^2} (q_{\lambda_n}(|\theta_*(s, \ell) + \theta(s, \ell)|) - q_{\lambda_n}(|\theta_*(s, \ell)|)), \quad \theta \in \mathcal{M}_1. \quad (7)$$

$U_n(\theta)$ is no other than $n^{-1} (Q_n(\theta_*) - Q_n(\theta_* + \theta))$ and is minimized at $\hat{\theta}_n - \theta_*^{(n)}$. We also introduce the conditional Kulback-Leibler divergence function

$$k^{(s)}(\theta_*, \theta) \stackrel{\text{def}}{=} \mathbb{E}_{\theta_*} \left(\int -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}})}{f_{\theta_*}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}})} \right) f_{\theta_*}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) du \right), \quad \theta \in \mathcal{M}_1. \quad (8)$$

By the concavity of the logarithm function, $k^{(s)}(\theta_*, \theta) \geq 0$. Finally, we define

$$k_n(\theta_*, \theta) \stackrel{\text{def}}{=} \sum_{s \in D_n} k^{(s)}(\theta_*, \theta), \quad \text{and} \quad k(\theta_*, \theta) \stackrel{\text{def}}{=} \sum_{s \in \mathcal{S}} k^{(s)}(\theta_*, \theta). \quad (9)$$

Clearly, $k_n(\theta_*, \theta)$ is nondecreasing in n and converges to $k(\theta_*, \theta)$. For any $\theta \in \mathcal{M}_1$ and $u \in \mathbb{X}$, we use (29) and (3) to verify that

$$\left| \log f_{\theta_* + \theta}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) - \log f_{\theta_*}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) \right| \leq \\ \left| \theta(s, s) B_0(u) + \sum_{\ell \in \mathcal{S} \setminus \{s\}} \theta(s, \ell) B(u, X_\ell) \right| + \left| \log Z_{n, \theta_* + \theta}^{(s)} - \log Z_{n, \theta_*}^{(s)} \right| \\ \leq C \left(|\theta(s, s)| + \sum_{\ell \in \partial_\theta s} |\theta(s, \ell)| \right),$$

for some finite constant C . This implies that $\sum_{s \in D_n} k^{(s)}(\theta_*, \theta) \leq C \|\theta\|_1 < \infty$ and proves that $k(\theta_*, \theta)$ is finite. Also notice that $\text{Argmin } k(\theta_*, \cdot)$ is nonempty and contains the null matrix 0 .

We study the consistency of $\hat{\theta}_n$ using epi-convergence methods as in Hess (1996). We review some definitions. For more on epi-convergence, we refer to Dal Maso (1993). Let (V, d) be a metric space and $\{f, f_n, n \geq 1\}$ be a sequence of functions defined on V , and taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$. The epi-limit inferior of $\{f_n, n \geq 1\}$ is the function

$$\text{li}_e f_n(x) = \sup_{k \geq 1} \liminf_{n \rightarrow \infty} \inf_{v \in B(x, k^{-1})} f_n(v),$$

where $B(x, \epsilon)$ denotes the open ball of V with center x and radius ϵ . We define similarly the epi-limit superior of $\{f_n, n \geq 1\}$ as

$$\text{ls}_e f_n(x) = \sup_{k \geq 1} \limsup_{n \rightarrow \infty} \inf_{v \in B(x, k^{-1})} f_n(v).$$

We say that f_n epi-converges (or Γ -converges) to f if $\text{ls}_e f_n(x) \leq f(x) \leq \text{li}_e f_n(x)$ for all $x \in V$.

Theorem 1.2. *Assume A1-2, (3) and suppose that $n^{-1}\lambda_n = o(1)$, as $n \rightarrow \infty$. Then almost surely, U_n epi-converges to $k(\theta_*, \cdot)$ in $(\mathcal{M}_1, \|\cdot\|_1)$.*

Proof. See Section 2.2.2. □

Epi-convergence is a very useful tool in the study of minimizers. The key result in that respect is as follows (using the notations of the above paragraph). If f_n epi-converges to f and $\{x_n, n \geq 1\}$ is such that $x_n \in \text{Argmin } f_n$, then if $x_n \rightarrow \bar{x}$ (in the metric space (V, d)), $\bar{x} \in \text{Argmin } f$. In order to make use of this result in our case, we need to impose additional conditions that ensure that $\hat{\theta}_n$ converges and that the limiting function $k(\theta_*, \cdot)$ admits a unique minimum.

For $s \in \mathcal{S}$ and $\theta \in \mathcal{M}$, define the infinite matrix $\rho_\theta^{(s)} \stackrel{\text{def}}{=} \{\rho_\theta^{(s)}(\ell, \ell'), \ell, \ell' \in \mathcal{S}\}$, where

$$\rho_\theta^{(s)}(\ell, \ell') \stackrel{\text{def}}{=} \mathbb{E}_\star [\text{Cov}_\theta(B(X_s, X_\ell), B(X_s, X_{\ell'}) | X_{\partial_\theta s})], \quad \ell, \ell' \in \mathcal{S}.$$

Corollary 1.3. *Suppose that the assumptions of Theorem 1.2 hold and also that for any $\theta \in \mathcal{M}$, any $s \in \mathcal{S}$, $\rho_\theta^{(s)}$ is a positive definite matrix. Let $\{\hat{\theta}_n, n \geq 1\}$ be a Borel measurable sequence of \mathcal{M}_1 such that $\hat{\theta}_n \in \text{Argmax } Q_n$. If $\{(\hat{\theta}_n - \theta_\star^{(n)}), n \geq 1\}$ is uniformly tight, as a random sequence of \mathcal{M}_1 , then $\|\hat{\theta}_n - \theta_\star^{(n)}\|_1$ converges in probability to zero.*

Proof. See Section 2.2.3. □

The tightness condition is needed but in general is difficult to check. Intuitively, the tightness of $\{(\hat{\theta}_n - \theta_\star^{(n)}), n \geq 1\}$ implies that the overall dependence between the missing nodes and the observed nodes is limited. We will not attempt to make this statement precise. We will rather study more precisely the connection between the missing nodes and the rate of convergence of $\hat{\theta}_n$. We assume that the following holds (see Section 1.2.1 for a discussion).

A3 Assume that there exist $\alpha_n, \alpha'_n > 0$ such that for all $\theta, \theta' \in \mathcal{M}^{(n)}$,

$$\sum_{s \in D_n} \mathbb{E}_{\theta_\star} \left[\text{Var}_{\theta'} \left(\sum_{\ell \in D_n} \theta(s, \ell) B(X_s, X_\ell) | X_{\partial_n s} \right) \right] \geq \alpha_n \|\theta\|_2^2,$$

$$\text{and } \mathbb{E}_\star^{1/2} \left[\left(\sum_{s \in D_n} \log \left(\frac{f_{\theta_\star^{(n)} + \theta}^{(s)}(X_s^{(1)} | X_{\partial_n s}^{(1)})}{f_{\theta_\star^{(n)}}^{(s)}(X_s^{(1)} | X_{\partial_n s}^{(1)})} \right) \right)^2 \right] \leq \alpha'_n \|\theta\|_2,$$

for all n large enough.

Let $\{a_n, n \geq 1\}$ be a sequence of positive numbers. We define $\mathcal{M}^{(n)}(a_n)$ the set of all finite $p_n \times p_n$ symmetric matrix θ such that

$$|\{(s, \ell) \in \underline{D}_n^2 : |\theta(s, \ell)| > 0\}| \leq a_n.$$

In other words, $\mathcal{M}^{(n)}(a_n)$ is the set of elements of $\mathcal{M}^{(n)}$ with sparsity a_n . We introduce $\Delta_n^{(c)} \stackrel{\text{def}}{=} \{s \in D_n : \partial s \setminus D_n \neq \emptyset\}$, the set of observed nodes that admit neighbors outside

D_n . We can think of $\Delta_n^{(c)}$ as the boundary of D_n . Let $\{\tau_n, n \geq 1\}$ be another sequence of positive numbers. We define $\mathcal{M}^{(n)}(a_n, \tau_n)$ as the set of all $\theta \in \mathcal{M}^{(n)}(a_n)$ such that

$$\left\{ \sum_{s \in \Delta_n^{(c)}} \left(\sum_{\ell \in D_n} |\theta(s, \ell)| \right)^2 \right\}^{1/2} \leq \tau_n \left\{ \sum_{(s, \ell) \in \underline{D}_n^2} |\theta(s, \ell)|^2 \right\}^{1/2}.$$

We relate the behavior of the estimator $\hat{\theta}_n$, to the class of functions $\mathcal{F}_{n, \delta} \stackrel{\text{def}}{=} \{m_{n, \theta}, \theta \in B_{n, \delta}\}$, where $B_{n, \delta} \stackrel{\text{def}}{=} \{\theta \in \mathcal{M}^{(n)}(a_n, \tau_n) : \|\theta\|_2 \leq \delta\}$, $\delta > 0$, and

$$m_{n, \theta}(x) = \sum_{s \in D_n} \log \left(\frac{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})} \right), \quad x \in \mathbb{X}^\infty. \quad (10)$$

It is clear that the size of the family $\mathcal{F}_{n, \delta}$ depends on the size of $B_{n, \delta}$. By the sparsity of $\mathcal{M}^{(n)}(a_n, \tau_n)$, and for $a_n \leq d_n/2$ (we recall that $d_n = p_n(p_n + 1)/2$, where $p_n = |D_n|$), we have

$$N(\epsilon, B_{n, \delta}, \|\cdot\|_2) \leq \left(\frac{c\delta d_n}{\epsilon a_n} \right)^{a_n}, \quad (11)$$

for some universal constant c , where $N(\epsilon, B_{n, \delta}, \|\cdot\|_2)$ denotes the ϵ -covering number of the set $B_{n, \delta}$ with respect to the ℓ^2 -norm on $\mathcal{M}_n(a_n, \tau_n)$. To see this, notice that the ϵ -covering number of the ℓ^2 -ball of \mathbb{R}^{a_n} with radius δ is bounded from above by $(3\delta/\epsilon)^{a_n}$. For $\theta \in \mathcal{M}^{(n)}(a_n, \tau_n)$, since the number of non-zeros entries of θ is bounded from above by a_n , there are at most $\binom{d_n}{a_n}$ ways of forming θ from a sequence of a_n non-zeros elements of \mathbb{R}^{a_n} . Thus $N(\epsilon, B_{n, \delta}, \|\cdot\|_2) \leq \binom{d_n}{a_n} (3\delta/\epsilon)^{a_n}$. By Stirling's formula,

$$\begin{aligned} \binom{d_n}{a_n} &\leq \sqrt{\frac{2c}{a_n}} \exp(-a_n \log(a_n/d_n) - (d_n - a_n) \log(1 - a_n/d_n)) \\ &\leq \exp(a_n (\log(d_n) - \log(a_n) + c)), \end{aligned}$$

for some finite constant c , which leads to (11). See also Vershynin (2009). Finally we introduce

$$b_n \stackrel{\text{def}}{=} \left\{ \sum_{s \in D_n} \left(\sum_{\ell \in \partial s \setminus D_n} |\theta_*(s, \ell)| \right)^2 \right\}^{1/2},$$

which measure the strength of the dependence between the missing nodes and the observed nodes. Our main result is as follows.

Theorem 1.4. *Assume (3), A3. Suppose that as $n \rightarrow \infty$, $a_n \sqrt{\log p_n} = O(\alpha'_n n^{1/2})$, and $\lambda_n = O(\sqrt{n \log p_n})$, and also that $(\hat{\theta}_n - \theta_*^{(n)}) \in \mathcal{M}_n(a_n, \tau_n)$ for all n large enough. Then $r_n \|\hat{\theta}_n - \theta_*^{(n)}\|_2 = O_p(1)$ as $n \rightarrow \infty$, where $r_n = \alpha_n \sqrt{n} / (\sqrt{a_n \log p_n} + \sqrt{n b_n \tau_n})$.*

Proof. See Section 2.3. □

The theorem implies that $\hat{\theta}_n$ is consistent in estimating $\theta_\star^{(n)}$ if $\tau_n b_n = o(\alpha_n)$. In the above result, we need to find a_n and τ_n that guarantee that $(\hat{\theta}_n - \theta_\star^{(n)}) \in \mathcal{M}_n(a_n, \tau_n)$ for all n large enough. Notice that for any $\theta \in \mathcal{M}^{(n)}$, we have trivially

$$\left\{ \sum_{s \in \Delta_n^{(c)}} \left(\sum_{\ell \in D_n} |\theta(s, \ell)| \right)^2 \right\}^{1/2} \leq 2 \sup_{\{s \in \Delta_n^{(c)}\}} |\{\ell \in D_n : |\theta(s, \ell)| > 0\}|^{1/2} \|\theta\|_2.$$

This means that any $\theta \in \mathcal{M}^{(n)}(a_n)$ also belongs to $\mathcal{M}^{(n)}(a_n, \tau_n)$, for $\tau_n = 2n_n^{1/2}$, where $n_n \stackrel{\text{def}}{=} \sup_{\{\theta \in \mathcal{M}^{(n)}(a_n)\}} \sup_{s \in \Delta_n^{(n)}} |\{\ell \in D_n : |\theta(s, \ell)| > 0\}|$. Therefore with this choice of τ_n , we can replace $\mathcal{M}^{(n)}(a_n, \tau_n)$ by $\mathcal{M}^{(n)}(a_n)$ in the above theorem. This leads to the following reformulation.

Corollary 1.5. *Assume (3), A3. Suppose that as $n \rightarrow \infty$, $a_n \sqrt{\log p_n} = O(\alpha'_n n^{1/2})$, and $\lambda_n = O(\sqrt{n \log p_n})$, and also that $(\hat{\theta}_n - \theta_\star^{(n)}) \in \mathcal{M}_n(a_n)$ for all n large enough. Then $r_n \|\hat{\theta}_n - \theta_\star^{(n)}\|_2 = O_p(1)$ as $n \rightarrow \infty$, where $r_n = \alpha_n \sqrt{n} / (\sqrt{a_n \log p_n} + \sqrt{n} b_n n_n^{1/2})$.*

If the penalty function is $q_\lambda(x) = \lambda x$, the extensive recent literature on lasso points to the fact that the estimator $\hat{\theta}_n$ is sparse and recovers the sparsity structure of the true network θ_\star (Banerjee et al. (2008); Meinshausen and Yu (2009); Guo et al. (2010)). This suggest that a_n can be taken proportional to the sparsity of $\theta_\star^{(n)}$. That is,

$$a_n \propto |\{(s, \ell) \in \underline{D}_n^2 : |\theta_\star(s, \ell)| > 0\}|.$$

Finally, we will point out that if $b_n = 0$, then there is no missing data problem. In that case, Theorem 1.4 yields a similar rate of convergence as in Guo et al. (2010).

1.2.1. Comment on A3. For $s \in D_n$, $\theta \in \mathcal{M}^{(n)}$, consider the following matrices $\rho_{n,\theta}^{(s)} \stackrel{\text{def}}{=} \{\rho_{n,\theta}^{(s)}(\ell, \ell'), \ell, \ell' \in D_n\}$ and $\bar{\rho}_{n,\theta} \stackrel{\text{def}}{=} \{\bar{\rho}_{n,\theta}(s, \ell; s', \ell'), (s, \ell), (s', \ell') \in D_n \times D_n\}$, where

$$\begin{aligned} \rho_{n,\theta}^{(s)}(\ell, \ell') &\stackrel{\text{def}}{=} \mathbb{E}_\star [\text{Cov}_\theta(B(X_s, X_\ell), B(X_s, X_{\ell'} | X_{\partial_n s}))], \text{ and } \bar{\rho}_{n,\theta}(s, \ell; s', \ell') \\ &\stackrel{\text{def}}{=} \mathbb{E}_\star [(B(X_s, X_\ell) - \mathbb{E}_\theta(B(X_s, X_\ell) | X_{\partial_n s})) (B(X_{s'}, X_{\ell'}) - \mathbb{E}_\theta(B(X_{s'}, X_{\ell'}) | X_{\partial_n s'}))]. \end{aligned}$$

It can be easily seen that if the smallest eigenvalue of $\rho_{n,\theta}^{(s)}$ is bounded from below by $\alpha_n > 0$, uniformly in s and θ , then the first part of A3 holds. Similarly, if the largest eigenvalue of $\bar{\rho}_{n,\theta}^{(s)}$ is bounded from above by $\alpha'_n < \infty$, uniformly in θ , then the second part of A3 holds.

In many practical examples, $B(x, y) = B_0(x)B_0(y)$, where B_0 is a nonnegative and bounded function. In that case, one can often check A3. Indeed, the matrix $\rho_{n,\theta}^{(s)}$ becomes $\rho_{n,\theta}^{(s)}(\ell, \ell') = \mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s,\ell'} \text{Var}_\theta(B(X_s) | X_{\partial_n s}))$, where $B_{s,\ell} = 1$ if $\ell = s$ and $B_{s,\ell} = B_0(X_\ell)$ otherwise. If there exists $c_n > 0$ such that $\text{Var}_\theta(B(X_s) | X_{\partial_n s}) \geq c_n$ for all $s \in D_n$ and all $\theta \in \mathcal{M}^{(n)}$, then the first part of A3 holds with $\alpha_n = c_n \alpha_{n,0}$, where $\alpha_{n,0} > 0$ is the smallest

of the eigenvalues of the matrices $\mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s',\ell'})$. Similarly,

$$\begin{aligned} \bar{\rho}_{n,\theta}(s, \ell; s', \ell') &= \mathbb{E}_\star [\bar{B}_{s,\ell} \bar{B}_{s',\ell'} (B(X_s) - \mathbb{E}_\theta(B(X_s)|X_{\partial_n s})) (B(X_{s'}) - \mathbb{E}_\theta(B(X_{s'})|X_{\partial_n s'}))] \\ &\leq C \mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s',\ell'}). \end{aligned}$$

Then the second part of A3 holds and we can take α'_n proportional to the largest eigenvalue of $\mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s',\ell'})$.

Example 1 (The auto-binomial and auto-logistic models). We consider here the particular case of the auto-binomial models which is an extension of the popular auto-logistic model. The auto-binomial model allows to model data where the available observation at each node can be seen as a number of successes over a given common number of trials. Fix $\kappa \geq 1$ the number of trials and set $\mathbf{X} = \{0, \dots, \kappa\}$. The interaction functions of the auto-binomial model are given by $A(u) = \binom{\kappa}{u}$, $B_0(u) = u$ and $B(u, v) = uv$. The particular case $\kappa = 1$ corresponds to the auto-logistic model. The modeling assumption here is that for any $s \in \mathcal{S}$,

$$X_s | X_{\partial_\theta s} = x_{\partial_\theta s} \sim \mathcal{B}(\kappa, \alpha_\theta^{(s)}(x_{\partial_\theta s})), \quad \text{where} \quad \log \left(\frac{\alpha_\theta^{(s)}(x_{\partial_\theta s})}{1 - \alpha_\theta^{(s)}(x_{\partial_\theta s})} \right) = \theta(s, s) + \sum_{\ell \in \mathcal{S} \setminus \{s\}} \theta(s, \ell) x_\ell.$$

In the above display, $\mathcal{B}(n, p)$ denotes the binomial distribution with parameters n, p . Now for $\theta \in \mathcal{M}^{(n)}$, $\text{Var}_\theta(B(X_s)|X_{\partial_n s}) = \kappa \alpha_{n,\theta}^{(s)} (1 - \alpha_{n,\theta}^{(s)})$, where $\alpha_{n,\theta}^{(s)}$ is given by $\alpha_{n,\theta}^{(s)} = \left(1 + \exp \left(-\theta(s, s) - \sum_{j \neq s, j=1}^{p_n} \theta(s, j) X_j \right)\right)^{-1}$. If we insist that $\sup_{s, \ell \in D_n} |\theta(s, \ell)| \leq K$, and that $\theta \in \mathcal{M}^{(n)}(a_n)$, then

$$\text{Var}_\theta(B(X_s)|X_{\partial_n s}) \geq 4^{-1} \kappa e^{-2K \mathbf{N}_n^{1/2}}, \quad s \in D_n, \theta \in \mathcal{M}^{(n)}(a_n),$$

where $\mathbf{N}_n \stackrel{\text{def}}{=} \sup_{\theta \in \mathcal{M}^{(n)}(a_n)} \sup_{s \in D_n} |\{\ell \in D_n : |\theta(s, \ell)| > 0\}|$, is the maximum degree in $\mathcal{M}^{(n)}(a_n)$. It follows that A3 holds with $\alpha_n = \alpha_{n,0} 4^{-1} \kappa e^{-2K \mathbf{N}_n^{1/2}}$, where $\alpha_{n,0} > 0$ is the smallest of the eigenvalues of the matrices $\mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s',\ell'})$; and α'_n can be take as proportional to the largest eigenvalue of $\mathbb{E}_\star (\bar{B}_{s,\ell} \bar{B}_{s',\ell'})$.

1.3. Monte Carlo Evidence. We consider the auto-logistic model where $\mathbf{X} = \{0, 1\}$, $A(x) = 0$, $B_0(x) = x$, and $B(x, y) = xy$. We work with the ℓ^1 penalty: $q_\lambda(x) = \lambda x$. With respect to the number of nodes, we consider two cases: $p = 50$ and $p = 80$. For each setting, we consider different values of n (the sample size) through the formula $n = a \log p / \beta^2$, where a is the number of non-zero elements of the true network structure that we choose to be approximately $1.3 * p$, and where β is chosen in the range $[0.3, 2.0]$ (for $p = 50$), and $[0.6, 2.0]$ (for $p = 80$).

We compare three settings. In Setting 1, there is no missing data, and the samples are generated exactly from (1), for $\theta = \theta_\star$ (we set up θ_\star such that $\theta_\star(s, \ell) \geq 0$ and we use Propp-Wilson's perfect sampler). In Setting 2 and 3, we generate the sample $(X_1^{(i)}, \dots, X_p^{(i)}, X_{p+1}^{(i)}, \dots, X_{p+r}^{(i)})$ from (1), for $\theta = \theta_\star$, and we retain only $(X_1^{(i)}, \dots, X_p^{(i)})$,

for $1 \leq i \leq n$. Thus there are r missing nodes. In Setting 2, we use $r = 8$, whereas in Setting 3, we set $r = 20$. Table 1 shows the corresponding values of b_n in each setting.

| | Setting 1, $r = 0$ | Setting 2, $r = 8$ | Setting 3, $r = 20$ |
|----------|--------------------|--------------------|---------------------|
| $p = 50$ | 0 | 1.8 | 4.41 |
| $p = 80$ | 0 | 1.8 | 3.6 |

TABLE 1. Values of b_n in each setting of the simulation.

Regardless of the data generation mechanism, we fit model (1) by ℓ^1 penalized pseudo-likelihood and compute the relative Mean Square Error $\mathbb{E}_\star \left(\|\hat{\theta} - \theta_\star\|_2 \right) / \|\theta_\star\|_2$, estimated from K replications of the estimator ($K = 50$). In Figure 1, we plot $\mathbb{E}_\star \left(\|\hat{\theta} - \theta_\star\|_2 \right) / \|\theta_\star\|_2$ as a function of β . As expected, the more missing data, the worst the estimator behaves. Notice that in Setting 2 (where $r = 8$), the loss of accuracy of the estimator is worst for $p = 50$ compared to $p = 80$, although we have the same value $b_n = 1.8$. This points to the fact that in the rate of convergence of $\hat{\theta}$, the factor b_n is modulated by a factor related to size of the problem as predicted in Theorem 1.4 (the term τ_n).

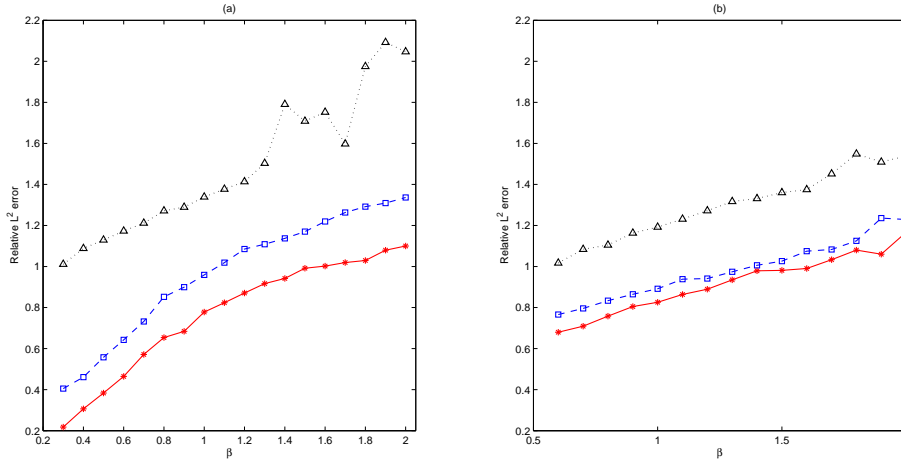


Figure 1: Relative MSE versus β , where star-line is Setting 1, square-line is Setting 2, triangle-line is Setting 3. (a) $p = 50$, (b) $p = 80$.

2. PROOFS

2.1. Some basic facts on infinite volume auto-models. We recall from Georgii (1988) some basic facts on Gibbs distributions. Let $(\mathbf{X}, \mathcal{E}, \rho)$ and \mathcal{S} as in Section 1.1. Let $(\Omega, \mathcal{F}) = (\mathbf{X}^{\mathcal{S}}, \mathcal{E}^{\mathcal{S}})$ be the product space equipped with the product Borel sigma-algebra. We will need few more notations. We denote by X_s the projection maps, that is, $X_s : (\Omega, \mathcal{F}) \rightarrow (\mathbf{X}, \mathcal{E})$ such that $X_s(\omega) = \omega_s$. For $A \subseteq \mathbf{X}$ and $\Delta \subset \mathcal{S}$, we denote by A^Δ the product set $\{(\omega_s)_{s \in \Delta}, \omega_s \in A\}$. We define $X_\Delta : (\Omega, \mathcal{F}) \rightarrow (\mathbf{X}^\Delta, \mathcal{E}^\Delta)$ as $X_\Delta(\omega) = \{\omega_s, s \in \Delta\}$ and we denote \mathcal{F}_Δ the sub σ -algebra of \mathcal{F} generated by the map X_U , $U \subseteq \Delta$, U finite.

For two disjoint subsets Λ, Δ of \mathcal{S} , if $u = \{x_i, i \in \Delta\}$ and $v = \{x_i, i \in \Lambda\}$ we write $uv = \{x_i, i \in \Delta \cup \Lambda\}$ for the concatenation of u, v .

For $\Delta \subset \mathcal{S}$ finite, we define the kernel ρ_Δ from $(\Omega, \mathcal{F}_{\mathcal{S} \setminus \Delta})$ to (Ω, \mathcal{F}) as follows

$$\rho_\Delta(\omega, A) \stackrel{\text{def}}{=} \left(\rho^\Delta \times \delta_{\omega_{\mathcal{S} \setminus \Delta}} \right) (A) = \rho^\Delta \left(\{u \in \mathbf{X}^\Delta : u\omega_{\mathcal{S} \setminus \Delta} \in A\} \right), \quad \omega \in \Omega, A \in \mathcal{F}.$$

In the above, δ_x is the Dirac mass at x and ρ^Δ denotes the product measure $\bigotimes_{s \in \Delta} \rho$ on $(\mathbf{X}^\Delta, \mathcal{E}^\Delta)$. This kernel is best understood through its operation on bounded functions. If $f : \Omega \rightarrow \mathbb{R}$ is a bounded measurable function and $\omega \in \Omega$, we have

$$\rho_\Delta f(\omega) \stackrel{\text{def}}{=} \int \rho_\Delta(\omega, dz) f(z) = \int_{\mathbf{X}^\Delta} f(u\omega_{\mathcal{S} \setminus \Delta}) \rho^\Delta(du).$$

For an infinite matrix $\theta : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ and Λ a finite subset of \mathcal{S} , we define $\pi_{\theta, \Lambda}$ a probability kernel from $(\Omega, \mathcal{F}_{\mathcal{S} \setminus \Lambda})$ to (Ω, \mathcal{F}) by

$$\pi_{\theta, \Lambda}(\omega, dz) = \frac{1}{Z_\theta(\omega)} \exp \{H_{\theta, \Lambda}(z)\} \rho_\Lambda(\omega, dz),$$

where for $z \in \Omega$,

$$H_{\theta, \Lambda}(z) = \sum_{s \in \Lambda} \left(A(z_s) + \theta(s, s)B_0(z_s) + \sum_{\ell \succeq s, \ell \neq s} \theta(s, \ell)B(z_s, z_\ell) \right).$$

The term $Z_\theta(\omega) \stackrel{\text{def}}{=} \rho_\Lambda H_{\theta, \Lambda}(\omega)$ is the normalizing constant. We write $\pi_\theta = \{\pi_{\theta, \Lambda}, \Lambda \subset \mathcal{S}, \Lambda \text{ finite}\}$ assuming that each kernel $\pi_{\theta, \Lambda}$ is well defined. If μ is a probability measure on (Ω, \mathcal{F}) , $h : \Omega \rightarrow \mathbb{R}$ a μ -integrable function and \mathcal{G} a sub-sigma-algebra of \mathcal{F} , we denote by $\mu(h|\mathcal{G})$ the conditional expectation of h given \mathcal{G} . An infinite volume auto-model is a probability measure μ_θ on (Ω, \mathcal{F}) that is consistent with the family $\pi_{\theta, \Lambda}$ in the sense that

$$\mu_\theta(f|\mathcal{F}_{\mathcal{S} \setminus \Lambda})(\cdot) = \int \pi_{\theta, \Lambda}(\cdot, dz) f(z), \quad \mu_\theta - a.s., \quad (12)$$

for any finite subset Λ of \mathcal{S} and any bounded measurable function $f : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$. Notice that (12) implies that $\mu_\theta \pi_{\theta, \Lambda} = \mu_\theta$, that is, each probability kernel in the family π_θ is invariant with respect to μ_θ . The probability measure μ_θ is an example of a Gibbs measure. We call a random variable $X = \{X_s, s \in \mathcal{S}\}$ with distribution μ_θ an auto-model random field with distribution μ_θ or with conditional specification π_θ . It is well known that given a conditional specification π_θ , a consistent Gibbs measure does not always exist and when it does, it is not necessarily unique. In the present case, infinite-volume auto-models exist. This follows for example from Georgii (1988), Theorem 4.23 (a).

Proposition 2.1. *Suppose that (3) holds and let $\theta : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}$ be an infinite matrix such that $\deg(s, \theta) < \infty$ for all $s \in \mathcal{S}$. Then the set of probability measure μ_θ that satisfies (12) is nonempty.*

2.2. Consistency: proof of Theorem 1.2. The theorem consists in showing that for almost all sample paths, the function U_n epi-converges to $k(\theta_\star; \cdot)$. It is obtained through a slight modification of Theorem 5.1 of Hess (1996).

2.2.1. Preliminaries. Let (V, d) be a Polish space with metric d and Borel sigma-algebra $\mathcal{B}(V)$. Let $\{g, f_n, n \geq 1\}$ be a sequence of real-valued functions defined on V . The next proposition states that if f_n can be written as $f_n = g_n + r_n$, where r_n converges to zero in an appropriate sense, then if g_n epi-converge to g , so does f_n . The proof is simple and is omitted. It also follows as a special case of Dal Maso (1993) Proposition 6.20.

Lemma 2.2. *Suppose that $f_n = g_n + r_n$, where $\{g, r_n, g_n, n \geq 1\}$ are real-valued functions defined on V , such that g_n epi-converges to g . Suppose that $|r_n(u)| \leq c(1 + d(u, 0))\alpha_n$ for all $u \in V$ and for some finite constant c , where $\alpha_n \rightarrow 0$. Then f_n epi-converges to g .*

For a real-valued function f on V , k integer, we define its Lipschitz approximation of order k as $f^{(k)}(u) \stackrel{\text{def}}{=} \inf_{v \in V} \{f(v) + kd(u, v)\}$. The Lipschitz approximation $f^{(k)}$ is a Lipschitz function (with Lipschitz coefficient k). For any $u \in V$ the sequence $\{f^{(k)}(u), k \geq 1\}$ is non-decreasing, upper bounded by $f(u)$ and if f itself is Lipschitz on V , $\sup_{k \geq 1} f^{(k)}(u) = f(u)$ (see e.g. Dal Maso (1993) Theorem 9.13).

Let (E, \mathcal{E}) be a measurable space. A more useful sigma-algebra to work with is $\hat{\mathcal{E}}$, the sigma-algebra of universally measurable subsets of E with respect to \mathcal{E} . $\hat{\mathcal{E}} = \cap_{\mu} \mathcal{E}_{\mu}$ where \mathcal{E}_{μ} is the μ -completion of \mathcal{E} with respect to a σ -finite measure μ on (E, \mathcal{E}) and where the intersection is over all σ -finite measures on (E, \mathcal{E}) . If $g : E \times V \rightarrow \mathbb{R}$ is a function, $x \in E$ and $k \geq 1$, we denote $g^{(k)}(x, \cdot)$ the Lipschitz approximations of order k of $g(x, \cdot)$. It is known (Hess (1996) Proposition 4.4) that if g is $\mathcal{E} \times \mathcal{B}(V)$ -measurable, then for any $k \geq 1$, $g^{(k)}$ is $\hat{\mathcal{E}} \times \mathcal{B}(V)$ -measurable. The following result is taken from Hess (1996) Proposition 3.4.

Proposition 2.3. *Let $\{g_n, n \geq 1\}$ be a sequence of $\mathcal{E} \times \mathcal{B}(V)$ -measurable real-valued functions satisfying the following assumptions. There exist a finite constant $c \in (0, \infty)$, $u_0 \in V$, such that $g_n(x, u_0) = 0$ for all $n \geq 1$, and*

$$\sup_{n \geq 1} \sup_{x \in E} |g_n(x, u) - g_n(x, v)| \leq cd(u, v) \quad u, v \in V, x \in E. \quad (13)$$

For $x \in E$, let $li_e g_n(x, \cdot)$ and $ls_e g_n(x, \cdot)$ be the epi-limit inferior and superior of the sequence $\{g_n(x, \cdot), n \geq 1\}$ respectively. Then for all $u \in V$,

$$li_e g_n(x, u) = \sup_{k \geq 1} \liminf_{n \rightarrow \infty} g_n^{(k)}(x, u), \quad \text{and} \quad ls_e g_n(x, u) = \sup_{k \geq 1} \limsup_{n \rightarrow \infty} g_n^{(k)}(x, u).$$

Proposition 2.4. *Let $\{g_n, n \geq 1\}$ be as in Proposition 2.3, and let $\{X_k, k \geq 1\}$ be a sequence of E -valued random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Define*

$$h_n(\omega, u) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n g_n(X_k(\omega), u), \quad n \geq 1, \omega \in \Omega, u \in V.$$

Suppose that there exists a Lipschitz function $\phi : V \rightarrow \mathbb{R}$ and $N \subseteq \Omega, \mathbb{P}(N) = 0$ such that for all $\omega \notin N$,

$$\lim_{n \rightarrow \infty} h_n(\omega, u) = \phi(u), \quad \text{for all } u \in V. \quad (14)$$

Then for all $\omega \notin N$, $ls_e h_n(\omega, u) \leq \phi(u)$, for all $u \in \mathbf{V}$, where $ls_e h_n(\omega, \cdot)$ is the epi-limit superior of the function $h_n(\omega, \cdot)$.

Proof. This result is part of Hess (1996) Theorem 5.1. We give the proof here for completeness. Fix $\omega \notin N$ and $u \in \mathbf{V}$. The Lipschitz property (13) of g_n transfers to h_n and by Proposition 2.3, $ls_e h_n(\omega, u) = \sup_{k \geq 1} \limsup_{n \rightarrow \infty} h_n^{(k)}(\omega, u)$. For any $k \geq 1$ there exists a sequence $\{v_p, p \geq 1\}$, $v_p = v_p(u, k) \in \mathbf{V}$ such that $\phi^{(k)}(u) = \inf_{p \geq 1} \{\phi(v_p) + kd(u, v_p)\}$. Then

$$\begin{aligned} \limsup_{n \rightarrow \infty} h_n^{(k)}(\omega, u) &= \limsup_{n \rightarrow \infty} \inf_{v \in \mathbf{V}} \{h_n(\omega, v) + kd(u, v)\} \leq \inf_{v \in \mathbf{V}} \limsup_{n \rightarrow \infty} \{h_n(\omega, v) + kd(u, v)\} \\ &\leq \inf_{p \geq 1} \limsup_{n \rightarrow \infty} \{h_n(\omega, v_p) + kd(u, v_p)\} = \inf_{p \geq 1} \{\phi(v_p) + kd(u, v_p)\} = \phi^{(k)}(u). \end{aligned}$$

Taking the supremum over k on both side gives the result. \square

We now consider the case where $\{X_k, k \geq 1\}$ is an i.i.d. sequence.

Proposition 2.5. *Let $\{g_n, n \geq 1\}$ be as in Proposition 2.3. Suppose that there exists a real-valued, $\mathcal{E} \times \mathcal{B}(\mathbf{V})$ -measurable function g such that*

$$\sup_{x \in E} |g(x, u) - g(x, v)| \leq cd(u, v) \quad u, v \in \mathbf{V}, \quad (15)$$

where c can be taken as in (13), and for any $(x, u) \in E \times \mathbf{V}$

$$\lim_{n \rightarrow \infty} |g_n(x, u) - g(x, u)| = 0. \quad (16)$$

Let $\{X_k, k \geq 1\}$ be a sequence of E -valued, i.i.d. random variables define on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We define, $\phi(u) \stackrel{\text{def}}{=} \mathbb{E}(g(X_1, u))$. Then there exists a \mathbb{P} -negligible subset N of Ω such that for any $u \in \mathbf{V}$ and $\omega \in \Omega \setminus N$,

$$h_n(\omega, u) \text{ epi-converges to } \phi(u), \text{ as } n \rightarrow \infty$$

where $h_n(\omega, u) \stackrel{\text{def}}{=} n^{-1} \sum_{k=1}^n g_n(X_k(\omega), u)$.

Proof. Notice that we can assume without any loss of generality that the constant c in (13) and (15) is smaller than 1. Otherwise simply divide g and g_n by $2c$, say. It follows from (13) that

$$\sup_{n \geq 1} \sup_{x \in E} |g_n(x, u)| \leq cd(u, u_0). \quad (17)$$

This implies that g_n is bounded in x and that $\phi_n(u) \stackrel{\text{def}}{=} \mathbb{E}(g_n(X_1, u))$ is well-defined and is uniformly bounded in n for each u . Now, since $g_n(x, u)$ converges pointwise to $g(x, u)$, we can then apply the Lebesgue dominated convergence theorem to conclude that $\phi_n(u) \rightarrow \phi(u)$ for each $u \in \mathbf{V}$. (15) implies also that ϕ is Lipschitz.

Furthermore, by the law of large numbers for arrays of independent random variables, for each $u \in \mathbf{V}$, there exists a measurable set $N_1(u) \subseteq \Omega$, $\mathbb{P}(N_1(u)) = 0$ such that for all $\omega \notin N_1(u)$, $\frac{1}{n} \sum_{k=1}^n (g_n(X_k(\omega), u) - \phi_n(u))$ converges to zero. Since $\phi_n(u) \rightarrow \phi(u)$ and using (13) and the Polish assumption, we conclude that there exists $N_1 \subseteq \Omega$, $\mathbb{P}(N_1) = 0$

such that for all $\omega \notin N_1$, $\lim_{n \rightarrow \infty} h_n(\omega, u) = \phi(u)$, for all $u \in V$. By Proposition 2.4, we obtain for all $\omega \notin N_1$, $\text{ls}_e h_n(\omega, u) \leq \phi(u)$, for all $u \in V$.

We will now show that there exists $N_2 \subseteq \Omega$, $\mathbb{P}(N_2) = 0$ such that for all $\omega \notin N_2$,

$$\liminf_{n \rightarrow \infty} h_n^{(k)}(\omega, u) \geq \mathbb{E} \left(g^{(k)}(X_1, u) \right), \quad \text{for all } u \in V, k \geq 1. \quad (18)$$

By Proposition 2.3, we can then deduce that for all $\omega \in \Omega \setminus N_2$, and for all $u \in V$, $\text{li}_e h_n(\omega, u) \geq \sup_{k \geq 1} \mathbb{E} \left(g^{(k)}(X_1, u) \right) = \lim_{k \rightarrow \infty} \mathbb{E} \left(g^{(k)}(X_1, u) \right) = \phi(u)$, by dominated convergence. And the result will be proved.

Let us show that (18) holds. Fix $u \in V$, $k \geq 1$ integer. Notice that $g_n^{(k)}(x, u) \leq g_n(x, u)$ and given the boundedness of g_n , we apply the law of large numbers to $g_n^{(k)}$ to conclude that there exists $N_2 \subseteq \Omega$, $\mathbb{P}(N_2) = 0$ such that for all $\omega \notin N_2$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} h_n^{(k)}(\omega, u) &\geq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g_n^{(k)}(X_i(\omega), u) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left(g_n^{(k)}(X_i(\omega), u) - \mathbb{E} \left(g_n^{(k)}(X_1, u) \right) \right) + \liminf_{n \rightarrow \infty} \mathbb{E} \left(g_n^{(k)}(X_1, u) \right) \\ &= \liminf_{n \rightarrow \infty} \mathbb{E} \left(g_n^{(k)}(X_1, u) \right). \end{aligned} \quad (19)$$

We obtain as a consequence of (13) that $|g_n(x, u)| \leq c d(u, u_0)$, for all n, x, u . Consequently, for any $v \in V$, and $k \geq 1$, $g_n(x, v) + k d(u, v) \geq -c d(v, u_0) + k d(u, v) \geq -c d(u, u_0)$. This shows that there exists a finite constant $C(u)$ (for example $c d(u, u_0)$) such that $g_n^{(k)}(x, u) + C(u) \geq 0$ for all $x \in E$. By Fatou's lemma, we deduce that

$$\liminf_{n \rightarrow \infty} \mathbb{E} \left(g_n^{(k)}(X_1, u) \right) \geq \mathbb{E} \left(\liminf_{n \rightarrow \infty} g_n^{(k)}(X_1, u) \right). \quad (20)$$

Fix $x \in E$. Given $\epsilon > 0$, we can find $v_0 = v_0(x, u, n, k, \epsilon) \in V$ such that $g_n^{(k)}(x, u) > g_n(x, v_0) + k d(u, v_0) - \epsilon$. Because of (13), $v_0 \in B(u, \epsilon/(1-c))$, where $B(x, r)$ is the ball of center x and radius r . Indeed, if $d(u, v) > \epsilon/(1-c)$, then $g_n(x, v) + k d(u, v) \geq g_n(x, u) + (k-c)d(u, v) \geq g_n(x, u) + \epsilon \geq g_n^{(k)}(x, u) + \epsilon$. Thus

$$\begin{aligned} g_n^{(k)}(x, u) &> g_n(x, v_0) + k d(u, v_0) - \epsilon = g(x, v_0) + k d(u, v_0) + (g(x, u) - g(x, v_0)) \\ &\quad + (g_n(x, u) - g(x, u)) + (g_n(x, v_0) - g(x, v_0)) - \epsilon \\ &\geq g^{(k)}(x, u) + (g_n(x, u) - g(x, u)) - (1-c)^{-1} \epsilon. \end{aligned}$$

Taking the \liminf as $n \rightarrow \infty$ on both side and letting $\epsilon \rightarrow 0$ together with (16) gives $\liminf_{n \rightarrow \infty} g_n^{(k)}(x, u) \geq g^{(k)}(x, u)$. Combining that with (19) and (20) yields (18). \square

2.2.2. Proof of Theorem 1.2. Write $U_n(\theta) = \bar{U}_n(\theta) + r_n(\theta)$, where

$$r_n(\theta) = n^{-1} \sum_{(s, \ell) \in \underline{D}_n^2} (q_{\lambda_n}(|\theta_\star(s, \ell) + \theta(s, \ell)|) - q_{\lambda_n}(|\theta_\star(s, \ell)|)).$$

Set $\mathcal{L}_0^{(n)}(\theta) \stackrel{\text{def}}{=} \{(s, \ell) \in \underline{D}_n^2 : \theta(s, \ell) \neq 0\}$. By the Mean Value Theorem and A1,

$$\begin{aligned} |r_n(\theta)| &= n^{-1} \lambda_n \left| \sum_{(s, \ell) \in \mathcal{L}_0^{(n)}(\theta)} \lambda_n^{-1} (q_{\lambda_n}(|\theta_*(s, \ell) + \theta(s, \ell)|) - q_{\lambda_n}(|\theta_*(s, \ell)|)) \right| \\ &\leq n^{-1} \lambda_n \sum_{(s, \ell) \in \mathcal{L}_0^{(n)}(\theta)} c (|\theta_*(s, \ell) + \theta(s, \ell)| - |\theta_*(s, \ell)|) \leq cn^{-1} \lambda_n \|\theta\|_1, \end{aligned}$$

for some finite constant c . Thus Lemma 2.2 applies and it is enough to show that almost surely, \bar{U}_n epi-converges to $k(\theta_*; \cdot)$.

To do so, we apply Proposition 2.5. Take $E = \mathsf{X}^{\mathcal{S}}$ with generic element $x = \{x(s), s \in \mathcal{S}\}$ and $V = \mathcal{M}_1$ and

$$g_n(x, \theta) = \sum_{s \in D_n} -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})} \right).$$

The limiting function g is given by

$$g(x, \theta) = \sum_{s \in \mathcal{S}} -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})} \right).$$

We have seen earlier that as a consequence of (Equation 29), $|g(x, \theta)| < \infty$. It is clear that g_n is a real-valued normal integrand, $g_n(x, 0) = 0$ and it follows also from (29) and (3) that

$$\sup_{x \in E} |g(x, \theta) - g(x, \theta')| + \sup_{n \geq 1} \sup_{x \in E} |g_n(x, \theta) - g_n(x, \theta')| \leq C \|\theta - \theta'\|_1,$$

for some finite constant C . Thus (13) and (15) hold. It remains to show (16).

Consider $x \in \mathsf{X}^{\mathcal{S}}$ and $\theta \in \mathcal{M}_1$. Since $\|\theta\|_1 < \infty$, for any $\epsilon > 0$, there exists a finite subset $\Lambda_\epsilon \subset \mathcal{S}$ such that $\sum_{(u, v) \notin \underline{\Lambda}_\epsilon^2} |\theta(u, v)| < \epsilon$. We have

$$\begin{aligned} |g_n(x, \theta) - g(x, \theta)| &\leq \sum_{s \in D_n} \left| -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})} \right) + \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})} \right) \right| \\ &\quad + \sum_{s \in \mathcal{S} \setminus D_n} \left| \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})} \right) \right|. \quad (21) \end{aligned}$$

We first deal with the second term on the right-hand side of (21). Fix $\epsilon > 0$. Take n large enough such that $\Lambda_\epsilon \subseteq D_n$. Then using again (29) and (3), we have

$$\sum_{s \in \mathcal{S} \setminus D_n} \left| \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{\mathcal{S} \setminus \{s\}})} \right) \right| \leq C \sum_{s \in \mathcal{S} \setminus D_n} \sum_{\ell \in \mathcal{S}} |\theta(s, \ell)| \leq C\epsilon.$$

The first term is obtained from

$$\begin{aligned} & \left(\log f_{\theta_* + \theta}^{(s)}(x_s | x_{S \setminus \{s\}}) - \log f_{\theta_*}^{(s)}(x_s | x_{S \setminus \{s\}}) \right) - \left(\log f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s}) - \log f_{\theta_*}^{(s)}(x_s | x_{\partial_n s}) \right) \\ &= \sum_{\ell \in S} \theta(s, \ell) B(x_s, x_\ell) - \int_0^1 dt \left\{ \int_X \sum_{\ell \in S} \theta(s, \ell) \bar{B}_{s, \ell}(u, x_\ell) f_{\theta_* + t\theta}^{(s)}(u | x_{S \setminus \{s\}}) \rho(du) \right\} \\ & \quad - \sum_{\ell \in D_n} \theta(s, \ell) B(x_s, x_\ell) + \int_0^1 dt \left\{ \int_X \sum_{\ell \in D_n} \theta(s, \ell) \bar{B}_{s, \ell}(u, x_\ell) f_{\theta_* + t\theta}^{(s)}(u | x_{\partial_n s}) \rho(du) \right\}, \end{aligned}$$

where $\bar{B}_{s, \ell}(x, y) = B_0(x)$ if $\ell = s$ and $\bar{B}_{s, \ell}(x, y) = B(x, y)$ otherwise. The above equality follows from Lemma 2.6. We use this to conclude that there exists a finite constant C such that

$$\begin{aligned} & \left| -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})} \right) + \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{S \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{S \setminus \{s\}})} \right) \right| \leq C \sum_{\ell \in S \setminus D_n} |\theta(s, \ell)| \\ & \quad + \sum_{\ell \in D_n} |\theta(s, \ell)| \int_0^1 dt \left| \int_X \bar{B}_{s, \ell}(u, x_\ell) \left(f_{\theta_* + t\theta}^{(s)}(u | x_{S \setminus \{s\}}) - f_{\theta_* + t\theta}^{(s)}(u | x_{\partial_n s}) \right) \rho(du) \right|. \end{aligned}$$

Taking the sum over $s \in D_n = \Lambda_\epsilon \cup D_n \setminus \Lambda_\epsilon$ we get

$$\begin{aligned} & \sum_{s \in D_n} \left| -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})} \right) + \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{S \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{S \setminus \{s\}})} \right) \right| \leq C \sum_{s \in D_n} \sum_{\ell \in S \setminus D_n} |\theta(s, \ell)| \\ & \quad + \sum_{s \in D_n} \sum_{\ell \in D_n} |\theta(s, \ell)| \int_0^1 dt \left| \int_X \bar{B}_{s, \ell}(u, x_\ell) \left(f_{\theta_* + t\theta}^{(s)}(u | x_{S \setminus \{s\}}) - f_{\theta_* + t\theta}^{(s)}(u | x_{\partial_n s}) \right) \rho(du) \right| \\ & \leq C\epsilon + \sum_{s \in \Lambda_\epsilon} \sum_{\ell \in S} |\theta(s, \ell)| \int_0^1 dt \left| \int_X \bar{B}_{s, \ell}(u, x_\ell) \left(f_{\theta_* + t\theta}^{(s)}(u | x_{S \setminus \{s\}}) - f_{\theta_* + t\theta}^{(s)}(u | x_{\partial_n s}) \right) \rho(du) \right|. \end{aligned}$$

For each s , the inner sum in the last term converges to 0 as $n \rightarrow \infty$. Since Λ_ϵ is finite, we conclude that

$$\lim_{n \rightarrow \infty} \sum_{s \in D_n} \left| -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{\partial_n s})}{f_{\theta_*}^{(s)}(x_s | x_{\partial_n s})} \right) + \log \left(\frac{f_{\theta_* + \theta}^{(s)}(x_s | x_{S \setminus \{s\}})}{f_{\theta_*}^{(s)}(x_s | x_{S \setminus \{s\}})} \right) \right| \leq C\epsilon.$$

It follows that (16) holds. Finally by conditioning on $X_{S \setminus \{s\}}$, we notice that

$$\begin{aligned} \mathbb{E}_{\theta_*} \left[-\log \left(\frac{f_{\theta_* + \theta}^{(s)}(X_s | X_{S \setminus \{s\}})}{f_{\theta_*}^{(s)}(X_s | X_{S \setminus \{s\}})} \right) \right] &= \mathbb{E}_{\theta_*} \left(\int -\log \left(\frac{f_{\theta_* + \theta}^{(s)}(u | X_{S \setminus \{s\}})}{f_{\theta_*}^{(s)}(u | X_{S \setminus \{s\}})} \right) f_{\theta_*}^{(s)}(u | X_{S \setminus \{s\}}) du \right) \\ &= k^{(s)}(\theta_*, \theta). \end{aligned}$$

The theorem is proved. \square

2.2.3. *Proof of Corollary 1.3.* Let us first show that $k(\theta_\star, \cdot)$ admits a unique minimum at 0. Since $k^{(s)}(\theta_\star, \cdot)$ is nonnegative, $k(\theta_\star, \theta) = 0$ implies that $k^{(s)}(\theta_\star, \theta) = 0$ for all $s \in \mathcal{S}$. We use Lemma 2.6 to write

$$\begin{aligned} -\log f_{\theta_\star + \theta}^{(s)}(X_s | X_{\mathcal{S} \setminus \{s\}}) + \log f_{\theta_\star}^{(s)}(X_s | X_{\mathcal{S} \setminus \{s\}}) = \\ -\sum_{\ell \in \mathcal{S}} \theta(s, \ell) \left(B(X_s, X_\ell) - \int_{\mathcal{X}} \bar{B}_{s, \ell}(u, X_\ell) f_{\theta_\star}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) du \right) \\ + \int_{\mathcal{X}} \sum_{\ell \in \mathcal{S}} \theta(s, \ell) \bar{B}_{s, \ell}(u, X_\ell) \int_0^1 dt \left(f_{\theta_\star + t\theta}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) - f_{\theta_\star}^{(s)}(u | X_{\mathcal{S} \setminus \{s\}}) \right) du. \end{aligned}$$

Taking the expectation on both side and using Lemma 2.6 again yields

$$\begin{aligned} k^{(s)}(\theta_\star, \theta) = \int_0^1 t dt \int_0^1 d\tau \mathbb{E}_\star \left[\text{Var}_{\theta_\star + t\tau\theta} \left(\sum_{\ell \in \mathcal{S}} \theta(s, \ell) \bar{B}_{s, \ell}(X_s, X_\ell) | X_{\mathcal{S} \setminus \{s\}} \right) \right] \\ = \int_0^1 t dt \int_0^1 d\tau \sum_{\ell, \ell' \in \mathcal{S}} \theta(s, \ell) \theta(s, \ell') \rho_{\theta_\star + t\tau\theta}^{(s)}(\ell, \ell'). \end{aligned}$$

Since $\rho_\theta^{(s)}$ is positive definite, $k^{(s)}(\theta_\star, \theta) = 0$ if and only if $\theta(s, \ell) = 0$ for all $\ell \in \mathcal{S}$.

Now, let $\epsilon > 0$. By tightness, there exists a compact subset \mathbf{K} of \mathcal{M}_1 such that $\sup_{n \geq 1} \check{\mathbb{P}}_\star \left((\hat{\theta}_n - \theta_\star^{(n)}) \notin \mathbf{K} \right) \leq \epsilon$. Therefore

$$\begin{aligned} \check{\mathbb{P}}_\star \left(\|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > \epsilon \right) \leq \epsilon + \check{\mathbb{P}}_\star \left((\hat{\theta}_n - \theta_\star^{(n)}) \in \mathbf{K}, \|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > \epsilon \right) \\ = \epsilon + \check{\mathbb{P}}_\star \left(\cup_{m \geq n} \left\{ (\hat{\theta}_m - \theta_\star^{(m)}) \in \mathbf{K}, \|\hat{\theta}_m - \theta_\star^{(m)}\|_1 > \epsilon \right\} \right). \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow \infty} \check{\mathbb{P}}_\star \left(\|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > \epsilon \right) \leq \epsilon + \check{\mathbb{P}}_\star \left(\left\{ (\hat{\theta}_n - \theta_\star^{(n)}) \in \mathbf{K}, \|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > \epsilon \right\} \text{ i.o.} \right).$$

Corollary 7.20 of Dal Maso (1993) and Theorem 1.2 imply that the probability on the rhs is zero. This ends the proof. \square

2.3. Rate of convergence: proof of Theorem 1.4. The proof of the theorem is adapted from Chapter 3.4 of van der Vaart and Wellner (1996) of the rate of convergence of M-estimators. Fix $\epsilon > 0$. Let $C, c_0 < \infty$ such that $\|B_0\|_\infty + \|B\|_\infty \leq C$ and $\sup_{\lambda > 0} \sup_{x > 0} q'_\lambda(x) \leq c_0$. Under the stated assumptions, $\alpha_n^{-1} r_n a_n^{1/2} \lambda_n n^{-1} = O(1)$, as $n \rightarrow \infty$. Therefore, we can take $M > 1$ large enough so that for all $n \geq 1$,

$$16c_0 \alpha_n^{-1} r_n a_n^{1/2} \lambda_n n^{-1} \leq 2^M, \quad \text{and} \quad 16 \sum_{j \geq M} 2^{-j} \leq \epsilon. \quad (22)$$

For $j \geq 1$, define $\Theta_{n,j} = \{\theta \in \mathcal{M}^{(n)}(a_n, \tau_n) : 2^{j-1} < r_n \|\theta\|_2 \leq 2^j\}$. Clearly we have,

$$\left\{ r_n \|\hat{\theta}_n - \theta_\star^{(n)}\|_2 > 2^M \right\} \subseteq \bigcup_{j \geq M} \left\{ \hat{\theta}_n - \theta_\star^{(n)} \in \Theta_{n,j} \right\}.$$

On the other hand, since U_n admits a minimum at $\hat{\theta}_n - \theta_\star^{(n)}$, almost surely, and $U_n(0) = 0$, it follows that $\{\hat{\theta}_n - \theta_\star^{(n)} \in \Theta_{n,j}\} \subseteq \{\inf_{\theta \in \Theta_{n,j}} U_n(\theta) \leq 0\}$. We conclude that

$$\check{\mathbb{P}}_\star \left(r_n \|\hat{\theta}_n - \theta_\star^{(n)}\|_2 > 2^M \right) \leq \sum_{j \geq M} \check{\mathbb{P}}_\star \left(\inf_{\theta \in \Theta_{n,j}} U_n(\theta) \leq 0 \right). \quad (23)$$

We recall that

$$\begin{aligned} U_n(\theta) &= n^{-1} \left(\bar{\ell}_n(\theta_\star^{(n)}) - \bar{\ell}_n(\theta_\star^{(n)} + \theta) \right) \\ &\quad + n^{-1} \sum_{(s,\ell) \in \underline{D}_n^2} (q_{\lambda_n}(|\theta_\star(s,\ell) + \theta(s,\ell)|) - q_{\lambda_n}(|\theta_\star(s,\ell)|)). \end{aligned}$$

Set $\mathcal{L}_0^{(n)}(\theta) = \{(s,\ell) \in \underline{D}_n^2 : \theta(s,\ell) \neq 0\}$. For $\theta \in \Theta_{n,j}$, and using the mean value theorem and A1,

$$\begin{aligned} &n^{-1} \sum_{(s,\ell) \in \underline{D}_n^2} (q_{\lambda_n}(|\theta_\star(s,\ell) + \theta(s,\ell)|) - q_{\lambda_n}(|\theta_\star(s,\ell)|)) \\ &= n^{-1} \lambda_n \sum_{(s,\ell) \in \mathcal{L}_0^{(n)}(\theta)} \lambda_n^{-1} (q_{\lambda_n}(|\theta_\star(s,\ell) + \theta(s,\ell)|) - q_{\lambda_n}(|\theta_\star(s,\ell)|)) \leq c_0 n^{-1} \lambda_n \sum_{(s,\ell) \in \mathcal{L}_0^{(n)}(\theta)} |\theta(s,\ell)| \\ &\leq c_0 n^{-1} \lambda_n a_n^{1/2} \|\theta\|_2 \leq c_0 n^{-1} \lambda_n a_n^{1/2} 2^j r_n^{-1}. \quad (24) \end{aligned}$$

Now, for $\theta \in \mathcal{M}^{(n)}$, $n^{-1} \left(\bar{\ell}_n(\theta_\star^{(n)}) - \bar{\ell}_n(\theta_\star^{(n)} + \theta) \right) = n^{-1} \sum_{i=1}^n m_{n,\theta}(X^{(i)}) = n^{-1} \sum_{i=1}^n \bar{m}_{n,\theta}(X^{(i)}) + M_n(\theta)$, where $\bar{m}_{n,\theta}(x) = m_{n,\theta}(x) - M_n(\theta)$, with $m_{n,\theta}$ as in (10), and

$$\begin{aligned} M_n(\theta) &= \sum_{s \in D_n} \sum_{\ell \in D_n} \theta(s,\ell) \mathbb{E}_\star \left[\int_0^1 dt \int \bar{B}_{s,\ell}(u, X_\ell) \left(f_{\theta_\star^{(n)}}^{(s)}(u|X_{\partial_n s}) - f_{\theta_\star^{(n)} + t\theta}^{(s)}(u|X_{\partial_n s}) \right) du \right] \\ &= \int_0^1 t dt \int_0^1 d\tau \sum_{s \in D_n} \mathbb{E}_\star \left[\text{Var}_{\theta_\star^{(n)} + t\tau\theta} \left(\sum_{\ell \in D_n} \theta(s,\ell) B(X_s, X_\ell) | X_{\partial_n s} \right) \right] \geq \frac{\alpha_n}{2} \|\theta\|_2^2, \end{aligned}$$

using Lemma 2.6 and A3. Notice that the first part of (22) implies that $\frac{\alpha_n}{4} 2^{2(j-1)} r_n^{-2} \geq c_0 2^j r_n^{-1} a_n^{1/2} \lambda_n n^{-1}$ whenever $j \geq M$. Therefore, using (24),

$$\inf_{\theta \in \Theta_{n,j}} \{U_n(\theta)\} \geq \inf_{\theta \in \Theta_{n,j}} \left\{ n^{-1} \sum_{i=1}^n \bar{m}_{n,\theta}(X^{(i)}) \right\} + \frac{\alpha_n}{4} 2^{2(j-1)} r_n^{-2},$$

and (23) becomes

$$\begin{aligned} \check{\mathbb{P}}_\star \left(r_n \|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > 2^M \right) &\leq \sum_{j \geq M} \check{\mathbb{P}}_\star \left(\sup_{\theta \in \Theta_{n,j}} \left| n^{-1/2} \sum_{i=1}^n \bar{m}_{n,\theta}(X^{(i)}) \right| \geq \frac{\alpha_n}{16} \frac{\sqrt{n} 2^{2j}}{r_n^2} \right) \\ &\leq \frac{16}{\alpha_n} \frac{r_n^2}{\sqrt{n}} \sum_{j \geq M} 2^{-2j} \left[\check{\mathbb{E}}_\star (\|\mathbb{G}_n\|_{\mathcal{F}_{n,j}}) + \sup_{\theta \in \Theta_{n,j}} \sqrt{n} |\mathbb{E}_{\theta_\star}(m_{n,\theta}(X)) - M_n(\theta)| \right], \quad (25) \end{aligned}$$

where \mathbb{G}_n is the empirical process associated to the family $\mathcal{F}_{n,j} = \mathcal{F}_{n,2^j r_n^{-1}}$: for $f \in \mathcal{F}_{n,j}$, $\mathbb{G}_n(f) = n^{-1/2} \sum_{i=1}^n (f(X^{(i)}) - \mathbb{E}_\star(f(X^{(1)})))$. And $\|\mathbb{G}_n\|_{\mathcal{F}_{n,j}} \stackrel{\text{def}}{=} \sup_{\theta \in \Theta_{n,j}} |\mathbb{G}_n(f)|$.

Using the fact that $\mathbb{E}_\star(B(X_s, X_\ell)|X_{S \setminus \{s\}}) = \int \bar{B}_{s,\ell}(u, X_\ell) f_{\theta_\star}^{(s)}(u|X_{S \setminus \{s\}}) du$, μ_\star -a.s., together with Lemma 2.6, we have

$$\begin{aligned} & |\mathbb{E}_{\theta_\star}(m_{n,\theta}(X)) - M_n(\theta)| \\ &= \left| \sum_{s \in D_n} \sum_{\ell \in D_n} \theta(s, \ell) \mathbb{E}_\star \left[\int_{\mathbf{X}} \bar{B}_{s,\ell}(X_s, X_\ell) \left(f_{\theta_\star}^{(s)}(u|X_{\partial s}) - f_{\theta_\star}^{(s)}(u|X_{\partial_n s}) \right) du \right] \right| \\ &\leq \sum_{s \in \Delta_n^{(c)}} \left(\sum_{\ell \in D_n} |\theta(s, \ell)| \right) \left(\sum_{\ell \in \partial s \setminus D_n} |\theta_\star(s, \ell)| \right) \leq b_n \tau_n \|\theta\|_2 \leq b_n \tau_n 2^j r_n^{-1}, \quad (26) \end{aligned}$$

where $\Delta_n^{(c)} = \{s \in D_n : \partial s \setminus D_n \neq \emptyset\}$. Notice that

$$|m_{n,\theta}(x)| \leq 2C \|\theta\|_1 \leq c \beta_{n,j}, \quad \text{for all } \theta \in \Theta_{n,j},$$

where $\beta_{n,j} = a_n^{1/2} 2^j r_n^{-1}$, for some finite constant c . Also for $\theta \in \Theta_{n,j}$, the second part of B3 yields

$$\mathbb{E}_\star^{1/2}(m_{n,\theta}^2(X)) \leq \alpha'_n \|\theta\|_2 \leq \delta_{n,j},$$

where $\delta_{n,j} = \alpha'_n 2^j r_n^{-1}$ for some finite constant c . By Lemma 3.4.2 of van der Vaart and Wellner (1996),

$$\check{\mathbb{E}}_\star(\|\mathbb{G}_n\|_{\mathcal{F}_{n,j}}) \leq c J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star})) \left(1 + \frac{c \beta_{n,j}}{\sqrt{n} \delta_{n,j}^2} J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star})) \right),$$

for some finite constant c , where $J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star}))$ is the bracketing integral of the family $\mathcal{F}_{n,j}$ defined as

$$J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star})) = \int_0^{\delta_{n,j}} \sqrt{1 + \log N_\square(\epsilon, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star}))} d\epsilon.$$

For any $\theta, \theta' \in \Theta_{n,j}$, $|m_{n,\theta}(x) - m_{n,\theta'}(x)| \leq c \|\theta - \theta'\|_1 \leq 2c a_n^{1/2} \|\theta - \theta'\|_2$, for all $x \in \mathbf{X}^\infty$. This Lipschitz property of the family $\mathcal{F}_{n,j}$, Theorem 2.7.11 of van der Vaart and Wellner (1996) and (11) imply that

$$\begin{aligned} J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star})) &\leq c a_n^{1/2} \int_0^{\delta_{n,j} a_n^{-1/2}/4c} \sqrt{1 + \log N(\epsilon, \Theta_{n,j}, \|\cdot\|_2)} d\epsilon \\ &\leq c \delta_{n,j} \sqrt{a_n \log \left(\frac{p_n}{a_n} \right)}, \end{aligned}$$

for some finite constant c . Under the assumption $a_n \sqrt{\log p_n} = O(\alpha'_n \sqrt{n})$, we obtain that $n^{-1/2} \beta_{n,j} \delta_{n,j}^{-2} J_\square(\delta_{n,j}, \mathcal{F}_{n,j}, L^2(\mu_{\theta_\star})) \leq c n^{-1/2} a_n \sqrt{\log p_n} / \alpha'_n = O(1)$. As the result, $\check{\mathbb{E}}_\star(\|\mathbb{G}_n\|_{\mathcal{F}_{n,j}}) \leq c \delta_{n,j} \sqrt{a_n \log p_n}$. Combined with (26) and (25) and the expression of r_n , it follows that $\check{\mathbb{P}}_\star(r_n \|\hat{\theta}_n - \theta_\star^{(n)}\|_1 > 2^M) \leq \epsilon c$ for some universal constant c . Since $\epsilon > 0$ is arbitrary, the theorem follows. \square

2.4. A comparison lemma.

Lemma 2.6. *Let (Y, \mathcal{A}, ν) be a measure space where ν is a finite measure. Let $g_1, g_2, f_1, f_2 : Y \rightarrow \mathbb{R}$ be bounded measurable functions. For $i \in \{1, 2\}$, define $Z_i = \int e^{g_i(y)} \nu(dy)$. For $t \in [0, 1]$, let $\bar{g}_t(\cdot) = tg_2(\cdot) + (1-t)g_1(\cdot)$ and $Z_t = \int_Y e^{\bar{g}_t(y)} \nu(dy)$. Let $\bar{f}_t : Y \rightarrow \mathbb{R}$ be such that $\bar{f}_0 = f_1$ and $\bar{f}_1 = f_2$. Suppose that $\frac{d}{dt} \bar{f}_t(y)$ exists for ν -almost all $y \in Y$ and $\sup_{t \in [0, 1], y \in Y} |\frac{d}{dt} \bar{f}_t(y)| < \infty$. Then*

$$\begin{aligned} \int f_2(y) e^{g_2(y)} Z_{g_2}^{-1} \nu(dy) - \int f_1(y) e^{g_1(y)} Z_{g_1}^{-1} \nu(dy) &= \int_0^1 dt \int_Y \left(\frac{d}{dt} \bar{f}_t(y) \right) e^{\bar{g}_t(y)} Z_t^{-1} \nu(dy) \\ &\quad + \int_0^1 dt \text{Cov}_t(\bar{f}_t(X), (g_2 - g_1)(X)), \end{aligned} \quad (27)$$

where $\text{Cov}_t(U_1(X), U_2(X))$ is the covariance between $U_1(X)$ and $U_2(X)$ assuming that $X \sim e^{\bar{g}_t(y)} Z_t^{-1}$.

Proof. Under the stated assumptions, the function $t \rightarrow \int_Y \bar{f}_t(y) e^{\bar{g}_t(y)} Z_t^{-1} \nu(dy)$ is differentiable under the integral sign and we have:

$$\int f_2(y) e^{g_2(y)} Z_{g_2}^{-1} \nu(dy) - \int f_1(y) e^{g_1(y)} Z_{g_1}^{-1} \nu(dy) = \int_0^1 \frac{d}{dt} \left(\int_Y \bar{f}_t(y) e^{\bar{g}_t(y)} Z_t^{-1} \nu(dy) \right) dt.$$

The identity follows by carrying the differentiation under the integral sign. \square

With the choice $\bar{f}_t(y) = tf_2(y) + (1-t)f_1(y)$, we get

$$\begin{aligned} \left| \int f_2(y) e^{g_2(y)} Z_{g_2}^{-1} \nu(dy) - \int f_1(y) e^{g_1(y)} Z_{g_1}^{-1} \nu(dy) \right| \\ \leq \|f_2 - f_1\|_\infty + 2(\|f_1\|_\infty + \|f_2\|_\infty) \|g_2 - g_1\|_\infty. \end{aligned} \quad (28)$$

We will also need the following particular case. For bounded measurable function $h_1, h_2 : Y \rightarrow \mathbb{R}$, we can take $f_i(y) \equiv \log \int e^{h_i(u)} \nu(du)$, $i = 1, 2$, $\bar{f}_t(y) \equiv \log \int e^{th_2(u) + (1-t)h_1(u)} \nu(du)$, and $g_1 = g_2$ in the lemma and get:

$$\begin{aligned} \log \int e^{h_2(y)} \nu(dy) - \log \int e^{h_1(y)} \nu(dy) &= \int_0^1 dt \left(\frac{d}{dt} \bar{f}_t \right) \\ &= \int_0^1 \int_Y (h_2(y) - h_1(y)) \frac{e^{th_2(u) + (1-t)h_1(u)}}{\int e^{th_2(u) + (1-t)h_1(u)} \nu(du)} \nu(dy). \end{aligned}$$

In particular,

$$\left| \log \int e^{h_2(y)} \nu(dy) - \log \int e^{h_1(y)} \nu(dy) \right| \leq \|h_2 - h_1\|_\infty. \quad (29)$$

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